An extension result for continuous valuations

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Abstract

We show, by a simple and direct proof, that if a bounded valuation on a directed complete partial order (dcpo) is the supremum of a directed family of simple valuations then it has a unique extension to a measure on the Borel σ-algebra of the dcpo with the Scott topology. It follows that every bounded and continuous valuation on a continuous domain can be extended uniquely to a Borel measure. The result also holds for σ-finite valuations, but fails for dcpo's in general.

1 Introduction

A valuation is a real valued function defined on the lattice of open sets of a given topological space that is modular, monotone and such that $\mu(\emptyset) = 0$ (see below). For domains, the problem of extending a (Scott) continuous valuation, i.e. one preserving directed suprema, to a Borel measure appeared in the context of probabilistic nondeterminism (see [17]). In [17] a proof was given for ω-algebraic domains but it contained a gap. Norberg established the result for σ-finite valuations on ω-continuous domains and gave applications in random set theory and a proof of the Daniell-Kolmogorov theorem for continuous lattices. For ω-continuous bounded complete domains the result followed from the work of Lawson [12] who showed that a continuous valuation defined on a distributive continuous lattice L has a unique extension to a regular Borel measure on L (on the Borel σ-algebra of its Lawson topology). Finally Jones and Plotkin [10,11], following the approach in [17], claimed the result for continuous domains, without presenting a correct proof. The gaps in [17] and [10] were pointed out in particular by O. Kirch and R. Tix.

Continuous valuations have in recent years played a crucial role in the domain theoretic approach to classical measure theory which has led to a new generalization of the Riemann integral with applications in fractal geometry [5,4,6].

In this extended abstract we give a short and direct proof for a general result that includes the domain theoretic cases mentioned above. We show that if a σ-finite valuation on a directed complete partial order equipped with the Scott topology, is the supremum of a directed family of simple valuations then it has a unique extension to a Borel measure As a corollary we have that every bounded and continuous valuation on a continuous domain can be extended uniquely to a Borel measure. We will only sketch the proofs of these results, the full proofs are given in the complete version of the paper [2].

We start by recalling some basic definitions. A topological space is a pair $(X, \Omega X)$ where X is a set and ΩX is a topology in X. We denote by $\mathcal{B}(X)$ the Borel $\sigma-$algebra of $(X, \Omega X)$. Any measure defined on $\mathcal{B}(X)$ is called a Borel measure. A Borel measure μ is finite if $\mu(X) < \infty$ and σ-**finite** if there exists a countable family $\{C_i\}_{i\in\mathbb{N}}$ of sets in $\mathcal{B}(X)$ such that $X = \bigcup_{i\in\mathbb{N}} C_i$ and $\mu(C_i) < \infty$ for all $i \in \mathbb{N}$. We say that a measure is normalised if $\mu(X) = 1$. Observe that every finite measure with $\mu(X) > 0$ can be normalised. For the measurable space $(X, \mathcal{B}(X))$ we denote by $\mathbf{M}X$ the set of all positive Borel measures bounded by 1 and by $\mathbf{M}^1X$ the set of all probability measures.

Our main reference for domain theory is [1]. Let $(P, \sqsubseteq)$ be a **partially ordered set** (poset). If $A \subseteq P$ we define $\downarrow A = \{x \in P \mid \exists a \in A.x \sqsubseteq a\}$. We often abbreviate $\downarrow\{a\}$ by $\downarrow a$. In a similar way we define $\uparrow A$ and $\uparrow a$. The set A is **lower** if $A = \downarrow A$ and **upper** if $A = \uparrow A$. A nonempty subset $A \subseteq P$ is said to be **directed** if for all $x, y \in A$ there exists $z \in A$ such that $x, y \sqsubseteq z$. A nonempty subset $T \subseteq P$ is called a **chain** if for all $x, y \in T$ we have $x \sqsubseteq y$ or $y \sqsubseteq x$. A poset P is a **directed complete partial order** (dcpo) if all directed subsets $A \subseteq P$ have a least upper bound (lub) denoted by $\bigsqcup^{\uparrow} A$. If $(P, \sqsubseteq)$ is a poset we define the **Scott topology** in P as follows: $O \subseteq P$ is Scott open if O is upper and for all directed subsets $A \subseteq P$ such that $\bigsqcup^{\uparrow} A$ exists, we have $\bigsqcup^{\uparrow} A \in O$ implies $O \cap A \neq \emptyset$. Note that $\downarrow a$ is Scott closed for all $a \in P$. The Scott topology will be denoted by $Scott(P)$. A dcpo with the Scott topology is a $\mathcal{T}_0$ space.

If x, y are elements of a dcpo D we say x approximates y or x is **way below** y, denoted by $x \ll y$ if whenever $y \sqsubseteq \bigsqcup^{\uparrow} A$, for a directed subset A, there exists $a \in A$ with $x \sqsubseteq a$. We have that $x \ll y$ implies $x \sqsubseteq y$. If $C \subseteq D$ we define $\twoheaduparrow C = \{x \in D \mid \exists c \in C.c \ll x\}$. We abbreviate $\twoheaduparrow\{c\}$ by $\twoheaduparrow c$. In a similar way we define $\twoheaddownarrow C$ and $\twoheaddownarrow c$. A subset $B \subseteq D$ is called a **basis** if for all $x \in D$ the set $B_x = B \cap \twoheaddownarrow x$ is directed and $x = \bigsqcup^{\uparrow} B_x$. D is a **continuous domain** if it has a basis and a ω-continuous domain if it has a countable basis. In a continuous domain D for all $C \subseteq D$ the set $\twoheaduparrow C$ is Scott open.

Given a topological space $(X, \Omega X)$, a **valuation** (see [3,12,11]) ν is a map $\nu : \Omega X \to [0, \infty]$ which satisfies:

- $\nu(\emptyset) = 0$ (strictness)
- $O_1 \subseteq O_2 \Rightarrow \nu(O_1) \leq \nu(O_2)$ (monotonicity)
- $\nu(O_1 \cup O_2) + \nu(O_1 \cap O_2) = \nu(O_1) + \nu(O_2)$ (modularity)

A valuation ν is said to be (Scott) continuous (see [12,11]) if for any $D \subseteq \Omega X$, which is directed with respect to $\subseteq$, we have $\nu(\bigcup D) = \sup_{O \in D} \nu(O)$. For any $a \in X$ we define the point valuation based at a as the function $\delta_a : \Omega X \to [0, \infty)$ such that

$$\delta_a(O) = \begin{cases} 1 & \text{if } a \in O \\ 0 & \text{otherwise} \end{cases}$$

A **simple valuation** ν is any finite linear combination $\sum_{i=1}^{n} r_i \delta_{a_i}$ of point valuations with $r_i \in \mathbb{R}^+ \setminus \{0\}$ for $i = 1, \ldots, n$. The set $\{a_1, a_2, \ldots, a_n\}$ is called the support of ν and is denoted by $|\nu|$. Simple valuations are always continuous. A simple valuation can be extended to a measure on the whole power set of X and its extension to a measure on $\mathcal{B}(X)$ is unique. For simplicity we will use the same name for a simple valuation and its extension.

The **probabilistic power domain** $\mathbf{P}X$ of a topological space $(X, \Omega X)$ is defined as the set of continuous valuations on X bounded by 1 with the following order: $\mu \sqsubseteq \nu$ iff $\mu(O) \leq \nu(O)$ for all $O \in \Omega X$. $(PX, \sqsubseteq)$ is a dcpo having the constant zero valuation as bottom element. For any directed subset $\{\nu_i\}_{i \in I}$ of PX we have $\nu = \bigsqcup^{\uparrow}_{i \in I} \nu_i$ is given by $\nu(O) = \sup_{i \in I} \nu_i(O)$ for all $O \in \Omega X$. We denote by $\mathbf{P}_s X$ the subset of simple valuations of $\mathbf{P}X$. The normalised probabilistic power domain of X is defined as $\mathbf{P}^1 X = \{\mu \in \mathbf{P}X \mid \mu(X) = 1\}$. If D is a dcpo with bottom element $\perp$ then $\mathbf{P}^1 D$ has $\delta_\perp$ as bottom element.

Let J be a directed set and L be a set. A **net** is a function $j \mapsto x_j : J \to L$. If L is a poset and $j \sqsubseteq k$ implies $x_j \sqsubseteq x_k$ for all $j, k \in J$ we say that the net is **monotone**. Nets will also be denoted as $\langle x_j \rangle_{j \in J}$. We will often use monotone nets for indexing directed subsets. Let $\langle x_j \rangle_{j \in J}$ be a net ranging over the real numbers. Recall that $\lim_{j \in J} x_j = l \in \mathbb{R}$ iff for all $\varepsilon > 0$ there exists $k \in J$ such that if $k \sqsubseteq j$ then $|x_j - l| < \varepsilon$. Let X be a set. A collection $\mathcal{S}$ of subsets of X is called a (Boolean) **semialgebra of sets** of X if: (i) $\emptyset, X \in \mathcal{S}$; (ii) $\mathcal{S}$ is closed under finite intersections; (iii) if $A \in \mathcal{S}$ then its complement A^c, can be expressed as a finite disjoint union of elements of $\mathcal{S}$.

2 Some auxiliary results

Lemma 2.1 *(Splitting lemma) Let X be a dcpo and $\nu_1 = \sum_{b \in |\nu_1|} r_b \delta_b, \nu_2 = \sum_{c \in |\nu_2|} s_c \delta_c \in \boldsymbol{P}^1 X$. Then $\nu_1 \sqsubseteq \nu_2$ if and only if there exists a function $f : |\nu_1| \times |\nu_2| \to [0, 1]$ such that for all $b \in |\nu_1|$ and $c \in |\nu_2|$*

(i) $\sum_{c \in |\nu_2|} f(b, c) = r_b$.

(ii) $\sum_{b \in |\nu_1|} f(b, c) = s_c$.

(iii) $f(b, c) > 0$ *implies* $b \sqsubseteq c$.

Proof. See [4, 3.1]. The original statement of the splitting lemma ([10, p. 83],[11, L. 9.2]) does not assume that the valuations are normalised, only asserts the sufficiency part of the implication and inequality in (ii). ∎

A set $T \subseteq D$ is called a crescent if there exist two open sets U and V such that $V \subseteq U$ and $T = U \backslash V$. We denote the set of all crescents of $(D, Scott(D))$ by $Cres(D)$. The next Proposition is a reformulation of Lemma 1 of [17] for dcpo's.

Proposition 2.2 *Let $(D, \sqsubseteq)$ be a dcpo.*

(i) *$Scott(D) \subseteq Cres(D)$.*

(ii) *$T \in Cres(D)$ if and only if there exist an open set A and a closed set B such that $T = A \cap B$.*

(iii) *If $T \in Cres(D)$ then*
- *If $x, y \in T$ and $x \sqsubseteq z \sqsubseteq y$ then $z \in T$.*
- *If $S \subseteq T$ is a directed subset then $\bigsqcup^{\uparrow} S \in T$.*
- *If $S \subseteq D$ is a directed subset and $\bigsqcup^{\uparrow} S \in T$ then there exists $d_0 \in S$ such that $d_0 \in T$.*

(iv) *$Cres(D)$ is a semialgebra.*

(v) *Let $\mu : Cres(D) \to [0, 1]$ be a function. Then μ has a unique extension to a probability measure on the Borel σ-algebra of $(D, Scott(D))$ if and only if μ is finitely additive, σ-subadditive and $\mu(D) = 1$.*

Lemma 2.3 *Let D be a continuous domain. Then D is ω-continuous if and only if $Scott(D)$ is second countable.*

A valuation ν on a topological space $(X, \Omega X)$ is called countably continuous if for all ω-chains $\{O_i\}_{i \in \mathbb{N}}$ in ΩX we have $\nu(\bigcup_{i \in \mathbb{N}} O_i) = sup_{i \in \mathbb{N}} \nu(O_i)$.

Lemma 2.4 *Let D be an ω-continuous domain. A valuation ν on D is continuous if and only if it is countably continuous.*

3 A generalized Splitting Lemma

Let $\{\nu_n\}_{n \in \mathbb{N}}$ be an ω-chain of normalised simple valuations on D. Consider a sequence of functions $f_i : |\nu_i| \times |\nu_{i+1}| \to [0, 1]$ such that f_i satisfies the conditions of the splitting lemma for ν_i and ν_{i+1}. Let $S = \prod_{i \in \mathbb{N}} |\nu_i|$. For any $E \subseteq S$ and $n \in \mathbb{N}$ define $E^n = \{(x_0, x_1, \ldots, x_n) \mid (x_m) \in E\}$. It follows that $S^n = \prod_{i=0}^{n} |\nu_i|$ for all $n \in \mathbb{N}$. For a given $s = (x_0, x_1, \ldots, x_n) \in S^n$ we define its weight by

$$w(s) = \nu_0(\{x_0\}) \prod_{0 \leq i < n} \frac{f_i(x_i, x_{i+1})}{\nu_i(\{x_i\})}$$

Each factor $\frac{f_i(x_i, x_{i+1})}{\nu_i(\{x_i\})} \leq 1$ can informally be interpreted as a conditional probability and, thus, it may be regarded as the probability of the transition from x_i to x_{i+1}. According to this interpretation, $w(s)$ will be the probability of the orbit s. Observe that $w(s) > 0$ implies $x_i \sqsubseteq x_{i+1}$ for all $i = 0, \ldots, n-1$. For any subset $C \subseteq S^n$ the weight of C is defined as $W(C) = \sum_{s \in C} w(s)$.

Lemma 3.1 *Let i_0 be a fixed integer such that $0 \leq i_0 \leq n$ and suppose $B \subseteq |\nu_{i_0}|$. Let $T \subseteq S^n$ be the set of sequences of S having their i_0^{th} component in B, i.e. $T = \prod_{0 \leq i < i_0} |\nu_i| \times B \times \prod_{i_0 < i \leq n} |\nu_i|$. Then $W(T) = \nu_{i_0}(B)$. In particular $W(S^n) = 1$ for all $n \in \mathbb{N}$.*

The Lemma's assertion corresponds to the intuitive idea that the probability of all orbits intersecting B is the same as the probability of B. Therefore W is a finite probability measure on each $S^n = \prod_{i=0}^{n} |\nu_i|$ for all $n \in \mathbb{N}$. $|\nu_n|$ can be seen as a topological space endowed with the discrete topology. Therefore S becomes a topological space with the product topology ΩS. We show that W can be extended to a probability measure P on the Borel σ-algebra $\mathcal{B}(S)$. This is done in a similar way to the usual construction of the product measure (see [7] for example). We know that a basis for the product topology is given by the set of finite intersection of cylinders

$$L = \{\bigcap_{i=1}^{m} \langle n_i, x_i \rangle \mid n_i \in \mathbb{N} \text{ and } x_i \in |\nu_{n_i}| \text{ for all } i = 1, \ldots m\} \cup \{S\}$$

where $\langle n, x \rangle = \{(x_m) \in S \mid x_n = x\}$. By Lemma 3.1 we know that

$$W(\langle n, x \rangle^m) = \begin{cases} W(S^m) = 1 & \text{for } m < n \\ \nu_n(\{x\}) & \text{for } m \geq n \end{cases}$$

For any member of L we define

$$P(\bigcap_{i=1}^{m} \langle n_i, x_i \rangle) = W(\bigcap_{i=1}^{m} \langle n_i, x_i \rangle)^{m_0} \text{ where } m_0 = \max\{n_1, n_2, \ldots, n_m\}$$

In particular $P(\langle n, x \rangle) = W(\langle n, x \rangle^n) = \nu_n(\{x\})$. Note that $S = \bigcup_{x \in |\nu_0|} \langle 0, x \rangle$ is a disjoint union of cylinders and $\sum_{x \in |\nu_0|} P(\langle 0, x \rangle) = \sum_{x \in |\nu_0|} \nu_0(\{x\}) = 1$. Therefore if S is a cylinder we have $P(S) = 1$ otherwise we define $P(S) = 1$.

Lemma 3.2 *P can be extended uniquely to a probability measure on the Borel σ-algebra $\mathcal{B}(S)$.*

For simplicity this extension will also be denoted by P. Now for all $E \subseteq S$ and $n \in \mathbb{N}$ define

$$[n, E] = \{(x_i) \in S \mid (x_0, x_1, \ldots, x_n) \in E^n\}$$

Notice that $[n, E] \supseteq [n+1, E] \supseteq E$ for all $n \in \mathbb{N}$. Let $cl(E)$ denote the topological closure of E.

Lemma 3.3 *For all $E \subseteq S$, $cl(E) = \bigcap_{n \in \mathbb{N}} [n, E]$.*

Let C be the set of chains in S, i.e. $C = \{(x_k) \in S \mid x_k \sqsubseteq x_{k+1}$ for all $k \in \mathbb{N}\}$.

Corollary 3.4 *The set C is closed and $P(C) = 1$.*

Proposition 3.5 *Let $\{\nu_n\}_{n \in \mathbb{N}}$ be an ω-chain in $\boldsymbol{P}_s^1 D$. Let $A_n \subseteq |\nu_n|$ be a pairwise disjoint sequence of sets and suppose for some fixed $\alpha > 0$, $\nu_n(A_n) >$*

α, for all $n \in \mathbb{N}$. Then there exists $(x_k) \in S$ with $x_k \sqsubseteq x_{k+1}$ for all $k \in \mathbb{N}$, intersecting infinitely many A_n's.

Proof. Let $C(A_n) = C \cap \bigcup_{x \in A_n} \langle n, x \rangle$, i.e. the set of chains in S intersecting A_n. By the last corollary we know that C is a Borel measurable set, therefore

$$\begin{aligned} P(C(A_n)) &= P(C \cap \textstyle\bigcup_{x \in A_n} \langle n, x \rangle) \\ &= P(\textstyle\bigcup_{x \in A_n} \langle n, x \rangle) \qquad \text{since } P(C) = 1 \\ &= \textstyle\sum_{x \in A_n} P(\langle n, x \rangle) \\ &= \textstyle\sum_{x \in A_n} \nu_n(\{x\}) \\ &= \nu_n(A_n) \\ &\geq \alpha \end{aligned}$$

Now observe that the subset of chains of C intersecting infinitely many times the A_n's is exactly $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} C(A_n) = \limsup C(A_n)$. By letting $E_m = \bigcup_{n \geq m} C(A_n)$ for all $m \in \mathbb{N}$ we get that (E_m) is a decreasing sequence of subsets of S with $P(E_m) \geq P(C(A_m)) \geq \alpha$ for all $m \in \mathbb{N}$. Therefore $P(\bigcap_{m \in \mathbb{N}} E_m) \geq \alpha > 0$. Since P is a measure we conclude $\bigcap_{m \in \mathbb{N}} E_m \neq \emptyset$. This last result also follows from extended versions of the Borel-Cantelli Lemma for nonindependent events (see [13, L. 3.5] or [14]). ∎

4 The extension results

We first prove the extension theorem in the case of normalised simple valuations. The proof recovers the original idea in [17]; proposition 3.5 corrects the gap. The general extension theorem follows easily from this.

Theorem 4.1 *Let D be a dcpo and $\langle \mu_j \rangle_{j \in J}$ be a directed subset of $\boldsymbol{P}^1_s D$ with $\mu = \sup_{j \in J} \mu_j$. Then μ has a unique extension to a measure on the Borel σ-algebra of $(D, Scott(D))$.*

Proof (Sketch) Since μ is the supremum of a directed family of continuous valuations it is easy to check that μ is also a continuous valuation and $\mu(D) = 1$. We will extend μ to the semialgebra $Cres(D)$ in order to apply part (v) of Proposition 2.2. For any $T = U \backslash V \in Cres(D)$ with $U, V \in Scott(D)$ and $V \subseteq U$ define $\overline{\mu}(T) = \mu(U) - \mu(V)$. We can check $\overline{\mu}$ is well defined and finitely additive. To see $\overline{\mu}$ is σ-subadditive, suppose $T \in Cres(D)$ is the union of a countable disjoint sequence (T_i) in $Cres(D)$ with $T = U \backslash V$ and $T_i = U_i \backslash V_i$ for all $i \in \mathbb{N}$. We show that $\overline{\mu}(T) \leq \sum_{i=1}^{\infty} \overline{\mu}(T_i)$. In order to derive a contradiction assume $\lim_m \sum_{i=1}^{m} \overline{\mu}(T_i) = \sum_{i=1}^{\infty} \overline{\mu}(T_i) < \overline{\mu}(T)$. This happens if and only if there exists some $\alpha > 0$ such that for all $m \in \mathbb{N}$ we have $\sum_{i=1}^{m} \overline{\mu}(T_i) \leq \overline{\mu}(T) - 2\alpha$. Therefore $2\alpha \leq \lim_j \mu_j(\bigcup_{i > m} T_i)$ This means that if we fix $m \in \mathbb{N}$ there exists $n_m \in J$ such that $n_m \sqsubseteq j$ implies $\mu_j(\bigcup_{i > m} T_i) > \alpha$.

Let $N_0 \in J$ be such that $\mu_{N_0}(T) > \alpha$. Observe that for all $j \in J$, the set

$|\mu_j| \cap T$ is finite. In particular since $|\mu_{N_0}|$ is finite, there exists $p_1 \in \mathbb{N}$ such that $|\mu_{N_0}| \cap T \subseteq \bigcup_{m \le p_1} T_m$. Since we are assuming $\lim_{j\in J} \mu_j(\bigcup_{m>p_1} T_m) \ge 2\alpha$, there exists $N_1 \in J$ such that $\mu_{N_1}(\bigcup_{m>p_1} T_m) > \alpha$. Because $\langle \mu_j \rangle_{j\in J}$ is directed we can also assume $\mu_{N_0} \sqsubseteq \mu_{N_1}$. In general, since $\left|\mu_{N_j}\right| \cap T \subseteq \bigcup_{m\le p_{j+1}} T_m$ is finite we can repeat the argument and find $N_{j+1} \in J$ and $p_{j+2} \in \mathbb{N}$ such that

- $\mu_{N_j} \sqsubseteq \mu_{N_{j+1}}$.
- $\left|\mu_{N_{j+1}}\right| \cap T \subseteq \bigcup_{m\le p_{j+2}} T_m$.
- $A_{j+1} = \left|\mu_{N_{j+1}}\right| \cap \bigcup_{p_{j+1}<m\le p_{j+2}} T_m$ with $\mu_{N_{j+1}}(A_{j+1}) > \alpha$.

Note that the sets A_j are pairwise disjoint for all $j \in \mathbb{N}$. Then $\{\mu_{N_n}\}_{n\in\mathbb{N}}$ is an ω-chain of normalised simple valuations with $\mu_{N_n}(A_n) > \alpha$. Applying Proposition 3.5 we know that there exists an ω-chain $\{x_k\}_{k\in\mathbb{N}}$ in D intersecting infinitely many A_n's. Since the sequence of terms intersecting the A_n's is also an ω-chain, without loss of generality, we can assume that each x_k belongs to one and only one A_n. Since T is a crescent and by part (iii) of Proposition 2.2 it follows that $\bigsqcup_{k\in\mathbb{N}} x_k \in T$. Hence there exists i_0 such that $\bigsqcup_{k\in\mathbb{N}} x_k \in T_{i_0}$. Again by part (iii) of Proposition 2.2 an infinite number of x_k will be in T_{i_0} and this is a contradiction since each x_k was chosen in a different T_i and the T_i are pairwise disjoint.

We conclude that $\overline{\mu}$ is σ-subadditive in $Cres(D)$. By part (v) of Proposition 2.2 we conclude that μ has a unique extension to a measure on the σ-algebra generated by $Scott(D)$. ∎

Corollary 4.2 *Let D be a dcpo and $\langle \mu_j \rangle_{j\in J}$ a directed set of simple valuations on D with $\mu = \sup_{j\in J} \mu_j$. If $\mu(D) < \infty$ then μ has a unique extension to a measure on the Borel σ-algebra of $(D, Scott(D))$.*

Corollary 4.3 *If D is a continuous domain then every bounded valuation has a unique extension to a measure on the Borel σ-algebra of $(D, Scott(D))$.*

Proof. If D is a continuous domain then the set of continuous valuations on D is also a continuous domain with a basis of simple valuations (see [10, Ch. 5]). Hence every bounded valuation is the supremum of a directed set of simple valuations and the conclusion follows from the last corollary. ∎

5 The σ-finite case

Next we present the extension result for σ-finite valuations. The results in this section are more general only in the case the dcpo does not have a least element. This is because under the assumptions of the following theorem if D has a bottom element $\perp$ then there exists a $i_0 \in \mathbb{N}$ such that $\perp \in O_{i_0}$ and this implies $O_{i_0} = D$. Therefore $\mu(D) < \infty$.

Theorem 5.1 *Let D be a dcpo. Let $\langle \mu_j \rangle_{j\in J}$ be a directed set of simple valuations on D with $\mu = \sup_{j\in J} \mu_j$. If $D = \bigcup_{i\in\mathbb{N}} O_i$ with $O_i \in Scott(D)$ and*

$\mu(O_i) < \infty$ then μ has a unique extension to a measure on the Borel σ-algebra of $(D, Scott(D))$.

Proof (Sketch) Without loss of generality we can assume $O_i \subseteq O_{i+1}$ for all $i \in \mathbb{N}$. As in the proof of Theorem 4.1 we will first extend μ to the semialgebra $Cres(D)$ and show that this extension is finitely additive and σ-subadditive. The idea is to extend μ to a finite measure on O_i and then take the supremum of such measures.

Let ν be a continuous valuation on D and $O \in Scott(D)$. The restriction of ν to O is defined as $\nu|_O(A) = \nu(A \cap O)$ for all $A \in Scott(D)$. This function is also a continuous valuation on D. If $\nu = \sum_{b \in |\nu|} r_b \delta_b$ is a simple valuation then $\nu|_O = \sum_{b \in |\nu| \cap O} r_b \delta_b$ is also a simple valuation. It is straightforward to check that $\langle \mu_j|_O \rangle_{j \in J}$ is a directed set of simple valuations on D and $\mu|_O = \sup_{j \in J} \mu_j|_O$. Notice that for all $i \in \mathbb{N}$, $\mu|_{O_i}(D) = \mu(O_i) < \infty$; hence by Corollary 4.2 it has a unique extension η_i to a Borel measure on $(D, Scott(D))$. Let $T = U \backslash V \in Cres(D)$ with $U, V \in Scott(D)$ and $V \subseteq U$. We define the extension of μ to $Cres(D)$ by $\overline{\mu}(T) = \sup_{i \in \mathbb{N}} \eta_i(T)$. We can check that $\overline{\mu}$ is an extension of μ, finitely additive and σ-subadditive, therefore μ has a unique extension to a Borel measure. ∎

In the general case where we only know $\mu(D) = \infty$. If D has a bottom element $\perp$ then the set $\{\perp\}$ is closed. Therefore $D' = D \backslash \{\perp\} \in Scott(D)$. Hence $\mu(D') < \infty$ or $\mu(D') = \infty$. In the first case it is straightforward to check that $\mu|_{D'} = \sup_{j \in J} \mu_j|_{D'}$ then by Corollary 4.2 $\mu|_{D'}$ can be extended uniquely to a Borel measure on D'. In order to extend this measure to the whole space D we only need to assign weight to $\{\perp\}$. Since a measure is finitely additive and we are assuming $\mu(D') < \infty$ the only possible choice is to have $\overline{\mu}(\{\perp\}) = \infty$. So if $\mu(D') < \infty$ the extension is unique. If $\mu(D') = \infty$ even if $\mu|_{D'}$ has a unique extension to a Borel measure on D', $\overline{\mu}(\{\perp\})$ can be assigned any nonnegative extended real value and this function will still be a measure. Therefore the extension to D is never unique.

We can now give a direct proof of a result by Norberg.

Corollary 5.2 *[15, Theorem 3.9] Let D be a continuous domain with a second countable Scott topology. A function $\mu : Scott(D) \to [0, \infty]$ that is finite on the set $\{U \in Scott(D) \mid U \ll D\}$ has a unique extension to a Borel measure if and only if it is a countably continuous valuation.*

Proof. The 'only if' part is immediate since the restriction of a Borel measure to the open sets gives a countably continuous valuation. The 'if' part follows from Lemmas 2.3, 2.4 and Theorem 5.1. ∎

6 The dcpo case

The question remains if the extension result holds in more general settings. We will give an example of a bounded continuous valuation on a dcpo which cannot be extended to a Borel measure. The following dcpo is defined in [9, ch. II, 1.9] (as an example of an non-sober dcpo). Consider $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with $(j,k) \sqsubseteq (m,n)$ iff either $j = m$ and $k \leq n$ or $n = \infty$ and $k \leq m$. It is easy to check that $(X, \sqsubseteq)$ is a dcpo and that every nonempty Scott open set contains all but a finite number of points (m, ∞).

Define the function $\nu : Scott(X) \to [0, \infty]$ by

$$\nu(O) = \begin{cases} 1 & \text{if } O \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Then ν is a modular function since the intersection of any pair of nonempty Scott open sets in X is again nonempty. Strictness, monotonicity and continuity are easily verified, therefore ν is a bounded continuous valuation. But ν cannot be extended to a Borel measure. In fact $O_n = (\bigcup_{j=0}^{n} \downarrow(j, \infty))^c$ is a decreasing sequence of open sets with $\bigcap_{n \in \mathbb{N}} O_n = \emptyset$ but $\lim_n \nu(O_n) = 1$. By the results in this paper ν also gives an example of a continuous valuation on a dcpo that is not the supremum of a directed family of simple valuations.

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